

# The maximal regularity operator on tent spaces

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En l'honneur des 60 ans de Michel Pierre

## Abstract

Recently, Auscher and Axelsson gave a new approach to non-smooth boundary value problems with  $L^2$  data, that relies on some appropriate weighted maximal regularity estimates. As part of the development of the corresponding  $L^p$  theory, we prove here the relevant weighted maximal estimates in tent spaces  $T^{p,2}$  for  $p$  in a certain open range. We also study the case  $p = \infty$ .

## 1 Introduction

Let  $-L$  be a densely defined closed linear operator acting on  $L^2(\mathbb{R}^n)$  and generating a bounded analytic semigroup  $(e^{-tL})_{t \geq 0}$ . We consider the maximal regularity operator defined by

$$\mathcal{M}_L f(t, x) = \int_0^t L e^{-(t-s)L} f(s, \cdot)(x) ds,$$

for functions  $f \in C_c(\mathbb{R}_+ \times \mathbb{R}^n)$ . The boundedness of this operator on  $L^2(\mathbb{R}_+ \times \mathbb{R}^n)$  was established by de Simon in [16]. The  $L^p(\mathbb{R}_+ \times \mathbb{R}^n)$  case, for  $1 < p < \infty$ , turned out, however, to be much more difficult. In [10], Kalton and Lancien proved that  $\mathcal{M}_L$  could fail to be bounded on  $L^p$  as soon as  $p \neq 2$ . The necessary and sufficient assumption for  $L^p$  boundedness was then found by Weis [17] to be a vector-valued strengthening of analyticity, called R-analyticity. As many differential operators  $L$  turn out to generate R-analytic semigroups, the  $L^p$  boundedness of  $\mathcal{M}_L$  has subsequently been successfully used in a variety of PDE situations (see [14] for a survey).

Recently, maximal regularity was used in a different manner as an important tool in [2], where a new approach to boundary value problems with  $L^2$  data for divergence form elliptic systems on Lipschitz domains, is developed. More precisely, in [2], the authors establish and use the boundedness of  $\mathcal{M}_L$  on weighted spaces  $L^2(\mathbb{R}_+ \times \mathbb{R}^n; t^\beta dt dx)$ , for certain values of  $\beta \in \mathbb{R}$ , under the additional assumption that  $L$  has bounded holomorphic functional calculus on  $L^2(\mathbb{R}^n)$ . This additional assumption was removed in [3, Theorem 1.3]. Here is the version when specializing the Hilbert space to be  $L^2(\mathbb{R}^n)$ .

**Theorem 1.1.** *With  $L$  as above,  $\mathcal{M}_L$  extends to a bounded operator on  $L^2(\mathbb{R}_+ \times \mathbb{R}^n; t^\beta dt dx)$  for all  $\beta \in (-\infty, 1)$ .*

The use of these weighted spaces is common in the study of boundary value problems, where they are seen as variants of the tent space  $T^{2,2}$  which occurs for  $\beta = -1$ , introduced by Coifman, Meyer and Stein in [6]. For  $p \neq 2$ , the corresponding spaces are weighted versions of the tent spaces  $T^{p,2}$ , which are defined, for parameters  $\beta \in \mathbb{R}$  and  $m \in \mathbb{N}$ , as the completion of  $C_c(\mathbb{R}_+ \times \mathbb{R}^n)$  with respect to

$$\|g\|_{T^{p,2,m}(t^\beta dt dy)} = \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{1_{B(x, t^{\frac{1}{m}})}(y)}{t^{\frac{n}{m}}} |g(t, y)|^2 t^\beta dy dt \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}},$$

the classical case corresponding to  $\beta = -1$ ,  $m = 1$ , and being denoted simply by  $T^{p,2}$ . The parameter  $m$  is used to allow various homogeneities, and thus to make these spaces relevant in the study of

differential operators  $L$  of order  $m$ . To develop an analogue of [2] for  $L^p$  data, we need, among many other estimates yet to be proved, boundedness results for the maximal operator  $\mathcal{M}_L$  on these tent spaces. This is the purpose of this note. Another motivation is well-posedness of non-autonomous Cauchy problems for operators with varying domains, which will be presented elsewhere. In the latter case,  $\mathcal{M}_L$  can be seen as a model of the evolution operators involved. However, as  $\mathcal{M}_L$  is an important operator on its own, we thought interesting to present this special case alone.

In Section 3 we state and prove the adequate boundedness results. The proof is based on recent results and methods developed in [9], building on ideas from [5] and [8]. In Section 2 we recall the relevant material from [9].

## 2 Tools

When dealing with tent spaces, the key estimate needed is a change of aperture formula, i.e., a comparison between the  $T^{p,2}$  norm and the norm

$$\|g\|_{T_\alpha^{p,2}} := \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{1_{B(x,\alpha t)}(y)}{t^n} |g(t,y)|^2 \frac{dy dt}{t} \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}},$$

for some parameter  $\alpha > 0$ . Such a result was first established in [6], building on similar estimates in [7], and analogues have since been developed in various contexts. Here we use the following version given in [9, Theorem 4.3].

**Theorem 2.1.** *Let  $1 < p < \infty$  and  $\alpha \geq 1$ . There exists a constant  $C > 0$  such that, for all  $f \in T^{p,2}$ ,*

$$\|f\|_{T^{p,2}} \leq \|f\|_{T_\alpha^{p,2}} \leq C(1 + \log \alpha) \alpha^{n/\tau} \|f\|_{T^{p,2}},$$

where  $\tau = \min(p, 2)$  and  $C$  depends only on  $n$  and  $p$ .<sup>1</sup>

Theorem 2.1 is actually a special case of the Banach space valued result obtained in [9]. Note, however, that it improves the power of  $\alpha$  appearing in the inequality from the  $n$  given in [6] to  $\frac{n}{\tau}$ . This is crucial in what follows, and has been shown to be optimal in [9].

Applying this to  $(t, y) \mapsto t^{\frac{m(\beta+1)}{2}} f(t^m, y)$  instead of  $f$ , we also have the weighted result, where

$$\|g\|_{T_\alpha^{p,2,m}(t^\beta dt dy)} = \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{1_{B(x,\alpha t^{\frac{1}{m}})}(y)}{t^{\frac{n}{m}}} |g(t,y)|^2 t^\beta dy dt \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}.$$

**Corollary 2.2.** *Let  $1 < p < \infty$ ,  $m \in \mathbb{N}$ ,  $\alpha \geq 1$ , and  $\beta \in \mathbb{R}$ . There exists a constant  $C > 0$  such that, for all  $f \in T^{p,2,m}(t^\beta dt dy)$ ,*

$$\|f\|_{T^{p,2,m}(t^\beta dt dy)} \leq \|f\|_{T_\alpha^{p,2,m}(t^\beta dt dy)} \leq C(1 + \log \alpha) \alpha^{n/\tau} \|f\|_{T^{p,2,m}(t^\beta dt dy)},$$

where  $\tau = \min(p, 2)$  and  $C$  depends only on  $n$  and  $p$ .

To take advantage of this result, one needs to deal with families of operators, that behave nicely with respect to tent norms. As pointed out in [9], this does not mean considering  $R$ -bounded families (which means  $R$ -analytic semigroups when one considers  $(tLe^{-tL})_{t \geq 0}$ ) as in the  $L^p(\mathbb{R}_+ \times \mathbb{R}^n)$  case, but tent bounded ones, i.e. families of operators with the following  $L^2$  off-diagonal decay, also known as Gaffney-Davies estimates.

**Definition 2.3.** A family of bounded linear operators  $(T_t)_{t \geq 0} \subset B(L^2(\mathbb{R}^n))$  is said to satisfy off-diagonal estimates of order  $M$ , with homogeneity  $m$ , if, for all Borel sets  $E, F \subset \mathbb{R}^n$ , all  $t > 0$ , and all  $f \in L^2(\mathbb{R}^n)$ :

$$\|1_E T_t 1_F f\|_2 \lesssim \left(1 + \frac{\text{dist}(E, F)^m}{t}\right)^{-M} \|1_F f\|_2.$$

In what follows  $\|\cdot\|_2$  denotes the norm in  $L^2(\mathbb{R}^n)$ .

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<sup>1</sup>only on pour éviter les confusions

As proven, for instance, in [4], many differential operators of order  $m$ , such as (for  $m = 2$ ) divergence form elliptic operators with bounded measurable complex coefficients, are such that  $(tLe^{-tL})_{t \geq 0}$  satisfies off-diagonal estimates of any order, with homogeneity  $m$ . This condition can, in fact, be seen as a replacement for the classical gaussian kernel estimates satisfied in the case of more regular coefficients.

### 3 Results

**Theorem 3.1.** *Let  $m \in \mathbb{N}$ ,  $\beta \in (-\infty, 1)$ ,  $p \in (\frac{2n}{n+m(1-\beta)}, \infty) \cap (1, \infty)$ , and  $\tau = \min(p, 2)$ . If  $(tLe^{-tL})_{t \geq 0}$  satisfies off-diagonal estimates of order  $M > \frac{n}{m\tau}$ , with homogeneity  $m$ , then  $\mathcal{M}_L$  extends to a bounded operator on  $T^{p,2,m}(t^\beta dtdy)$ .*

*Proof.* The proof is very much inspired by similar estimates in [5] and [9]. Let  $f \in \mathcal{C}_c(\mathbb{R}_+ \times \mathbb{R}^n)$ . Given  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ , and  $j \in \mathbb{Z}_+$ , we consider

$$C_j(x, t) = \begin{cases} B(x, t) & \text{if } j = 0, \\ B(x, 2^j t) \setminus B(x, 2^{j-1} t) & \text{otherwise.} \end{cases}$$

We write  $\|\mathcal{M}_L f\|_{T^{p,2}} \leq \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} I_{k,j} + \sum_{j=0}^{\infty} J_j$  where

$$I_{k,j} = \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{1_{B(x, t \frac{1}{m})}(y)}{t^{\frac{n}{m}}} \left| \int_{2^{-k-1}t}^{2^{-k}t} Le^{-(t-s)L} (1_{C_j(x, 4t \frac{1}{m})} f(s, \cdot))(y) ds \right|^2 t^\beta dy dt \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}},$$

$$J_j = \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{1_{B(x, t \frac{1}{m})}(y)}{t^{\frac{n}{m}}} \left| \int_{\frac{t}{2}}^t Le^{-(t-s)L} (1_{C_j(x, 4s \frac{1}{m})} f(s, \cdot))(y) ds \right|^2 t^\beta dy dt \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}.$$

Fixing  $j \geq 0$ ,  $k \geq 1$  we first estimate  $I_{k,j}$  as follows. For fixed  $x \in \mathbb{R}^n$ ,

$$\begin{aligned} & \int_0^\infty \int_{B(x, t \frac{1}{m})} \left| \int_{2^{-k-1}t}^{2^{-k}t} Le^{-(t-s)L} (1_{C_j(x, 4t \frac{1}{m})} f(s, \cdot))(y) ds \right|^2 t^{\beta - \frac{n}{m}} dy dt \\ & \leq \int_0^\infty \int_{B(x, t \frac{1}{m})} \left( \int_{2^{-k-1}t}^{2^{-k}t} \left| (t-s) Le^{-(t-s)L} (1_{C_j(x, 4t \frac{1}{m})} f(s, \cdot))(y) \right| \frac{ds}{t-s} \right)^2 t^{\beta - \frac{n}{m}} dy dt \\ & \lesssim \int_0^\infty \int_{2^{-k-1}t}^{2^{-k}t} 2^{-k} t \left( \int_{B(x, t \frac{1}{m})} \left| (t-s) Le^{-(t-s)L} (1_{C_j(x, 4t \frac{1}{m})} f(s, \cdot))(y) \right|^2 dy \right) t^{\beta - \frac{n}{m} - 2} ds dt \\ & \lesssim \int_0^\infty \int_{2^{-k-1}t}^{2^{-k}t} 2^{-k} \left( 1 + \frac{2^{jm}t}{t-s} \right)^{-2M} \|1_{B(x, 2^{j+2}t \frac{1}{m})} f(s, \cdot)\|_2^2 t^{\beta - \frac{n}{m} - 1} ds dt \\ & \lesssim 2^{-k} 2^{-2jmM} \int_0^\infty \left( \int_{2^k s}^{2^{k+1}s} t^{\beta - \frac{n}{m} - 1} dt \right) \|1_{B(x, 2^{j+\frac{k}{m}+3} s \frac{1}{m})} f(s, \cdot)\|_2^2 ds \\ & \lesssim 2^{-k(\frac{n}{m}+1-\beta)} 2^{-2jmM} \int_0^\infty \|1_{B(x, 2^{j+\frac{k}{m}+3} s \frac{1}{m})} f(s, \cdot)\|_2^2 s^{\beta - \frac{n}{m}} ds. \end{aligned}$$

In the second inequality, we use Cauchy-Schwarz inequality for the integral with respect to  $t$ , the fact that  $t-s \sim t$  for  $s \in \cup_{k \geq 1} [2^{-k-1}t, 2^{-k}t] \subset [0, \frac{t}{2}]$  and Fubini's theorem to exchange the integral in  $t$  and the integral in  $y$ . The next inequality follows from the off-diagonal estimate verified by  $(t-s)Le^{-(t-s)L}$  and again the fact that  $t-s \sim t$ . By Corollary 2.2 this gives

$$I_{k,j} \lesssim (j+k) 2^{-k(\frac{1}{2}(\frac{n}{m}+1-\beta) - \frac{n}{m\tau})} 2^{-j(mM - \frac{n}{\tau})} \|f\|_{T^{p,2,m}(t^\beta dtdy)},$$

where  $\tau = \min(p, 2)$ . It follows that  $\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} I_{k,j} \lesssim \|f\|_{T^{p,2,m}(t^\beta dt dy)}$  since  $M > \frac{n}{m\tau}$  and  $\frac{n}{m} + 1 - \beta > \frac{2n}{m\tau}$  (Note that for  $p \geq 2$ , this requires  $\beta < 1$ ).

We now turn to  $J_0$  and remark that  $J_0 \leq \left( \int_{\mathbb{R}^n} J_0(x)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}$ , where

$$J_0(x) = \int_0^\infty \int_{\mathbb{R}^n} \left| \int_{\frac{t}{2}}^t L e^{-(t-s)L} (g(s, \cdot))(y) ds \right|^2 t^{\beta - \frac{n}{m}} dy dt$$

with  $g(s, y) = 1_{B(x, 4s^{\frac{1}{m}})}(y) f(s, y)$ . The inside integral can be rewritten as

$$\mathcal{M}_L g(t, \cdot) - e^{-\frac{t}{2}L} \mathcal{M}_L g\left(\frac{t}{2}, \cdot\right).$$

As  $\mathcal{M}_L$  is bounded on  $L^2(\mathbb{R}_+ \times \mathbb{R}^n; t^{\beta - \frac{n}{m}} dy dt)$  by Theorem 1.1 and  $(e^{-tL})_{t \geq 0}$  is uniformly bounded on  $L^2(\mathbb{R}^n)$ , we get

$$J_0(x) \lesssim \int_0^\infty \left\| 1_{B(x, 4s^{\frac{1}{m}})} f(s, \cdot) \right\|_2^2 s^{\beta - \frac{n}{m}} ds.$$

We finally turn to  $J_j$ , for  $j \geq 1$ . For fixed  $x \in \mathbb{R}^n$ ,

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^n} \left| 1_{B(x, t^{\frac{1}{m}})}(y) \int_{\frac{t}{2}}^t L e^{-(t-s)L} (1_{C_j(x, 4s^{\frac{1}{m}})} f(s, \cdot))(y) ds \right|^2 t^{\beta - \frac{n}{m}} dy dt \\ & \leq \int_0^\infty \int_{\mathbb{R}^n} 1_{B(x, t^{\frac{1}{m}})}(y) \left( \int_{\frac{t}{2}}^t |(t-s) L e^{-(t-s)L} (1_{C_j(x, 4s^{\frac{1}{m}})} f(s, \cdot))(y)| \frac{ds}{t-s} \right)^2 t^{\beta - \frac{n}{m}} dy dt \\ & \lesssim \int_0^\infty \int_{\mathbb{R}^n} 1_{B(x, t^{\frac{1}{m}})}(y) \int_{\frac{t}{2}}^t |(t-s) L e^{-(t-s)L} (1_{C_j(x, 4s^{\frac{1}{m}})} f(s, \cdot))(y)|^2 \frac{ds}{(t-s)^2} t^{\beta - \frac{n}{m} + 1} dy dt \\ & \lesssim \int_0^\infty \int_{\frac{t}{2}}^t (t-s)^{-2} \left( 1 + \frac{2^{jm}t}{t-s} \right)^{-2M} \left\| 1_{B(x, 2^{j+2}s^{\frac{1}{m}})} f(s, \cdot) \right\|_2^2 s^{\beta - \frac{n}{m} + 1} ds dt \\ & \lesssim 2^{-jm(2M-2)} \int_0^\infty \left( \int_s^{2s} s(t-s)^{-2} \left( 1 + \frac{2^{jm}t}{t-s} \right)^{-2} dt \right) \left\| 1_{B(x, 2^{j+2}s^{\frac{1}{m}})} f(s, \cdot) \right\|_2^2 s^{\beta - \frac{n}{m}} ds \\ & \lesssim 2^{-2jmM} \int_0^\infty \left\| 1_{B(x, 2^{j+2}s^{\frac{1}{m}})} f(s, \cdot) \right\|_2^2 s^{\beta - \frac{n}{m}} ds, \end{aligned}$$

where we have used Cauchy-Schwarz inequality in the second inequality, the off-diagonal estimates and the fact that  $s \leq t$  in the third, Fubini's theorem and the fact that  $s \geq \frac{t}{2}$  in the fourth, and the change of variable  $\sigma = \frac{t}{t-s}$  in the last. An application of Corollary 2.2, then gives

$$J_j \lesssim 2^{-jmM} j 2^{j\frac{n}{\tau}} \|f\|_{T^{p,2,m}(t^\beta dt dy)} = j 2^{-j(mM - \frac{n}{\tau})} \|f\|_{T^{p,2,m}(t^\beta dt dy)},$$

and the proof is concluded by summing the estimates.  $\square$

An end-point result holds for  $p = \infty$ . In this context the appropriate tent space consists of functions such that  $|g(t, x)|^2 \frac{dx dt}{t}$  is a Carleson measure, and is defined as the completion of the

space  $\mathcal{C}_c(\mathbb{R}_+ \times \mathbb{R}^n)$  with respect to

$$\|g\|_{T^{\infty,2}}^2 = \sup_{(x,r) \in \mathbb{R}^n \times \mathbb{R}_+} r^{-n} \int_{B(x,r)} \int_0^r |g(t,x)|^2 \frac{dx dt}{t}.$$

We also consider the weighted version defined by

$$\|g\|_{T^{\infty,2,m}(t^\beta dt dy)}^2 := \sup_{(x,r) \in \mathbb{R}^n \times \mathbb{R}_+} r^{-\frac{n}{m}} \int_{B(x,r^{\frac{1}{m}})} \int_0^r |g(t,x)|^2 t^\beta dx dt.$$

**Theorem 3.2.** *Let  $m \in \mathbb{N}$ , and  $\beta \in (-\infty, 1)$ . If  $(tLe^{-tL})_{t \geq 0}$  satisfies off-diagonal estimates of order  $M > \frac{n}{2m}$ , with homogeneity  $m$ , then  $\mathcal{M}_L$  extends to a bounded operator on  $T^{\infty,2,m}(t^\beta dt dy)$ .*

*Proof.* Pick a ball  $B(z, r^{\frac{1}{m}})$ . Let

$$I^2 = \int_{B(z, r^{\frac{1}{m}})} \int_0^r |(\mathcal{M}_L f)(t, x)|^2 t^\beta dx dt.$$

We want to show that  $I^2 \lesssim r^{\frac{n}{m}} \|f\|_{T^{\infty,2}(t^\beta dt dy)}^2$ . We set

$$I_j^2 = \int_{B(x, r^{\frac{1}{m}})} \int_0^r |(\mathcal{M}_L f_j)(t, x)|^2 t^\beta dx dt$$

where  $f_j(s, x) = f(s, x) 1_{C_j(z, 4r^{\frac{1}{m}})}(x) 1_{(0,r)}(s)$  for  $j \geq 0$ . Thus by Minkowsky inequality,  $I \leq \sum I_j$ . For  $I_0$  we use again Theorem 1.1 which implies that  $\mathcal{M}_L$  is bounded on  $L^2(\mathbb{R}_+ \times \mathbb{R}^n, t^\beta dx dt)$ . Thus

$$I_0^2 \lesssim \int_{B(z, 4r^{\frac{1}{m}})} \int_0^r |f(t, x)|^2 t^\beta dx dt \lesssim r^{\frac{n}{m}} \|f\|_{T^{\infty,2,m}(t^\beta dt dy)}^2.$$

Next, for  $j \neq 0$ , we proceed as in the proof of Theorem 3.1 to obtain

$$\begin{aligned} I_j^2 &\lesssim \sum_{k=1}^{\infty} \int_0^r \int_{2^{-k-1}t}^{2^{-k}t} 2^{-k} t \left(1 + \frac{2^{jm}r}{t-s}\right)^{-2M} \|f_j(s, \cdot)\|_{L^2}^2 t^{\beta-2} ds dt \\ &\quad + \int_0^r \int_{\frac{t}{2}}^t t(t-s)^{-2} \left(1 + \frac{2^{jm}r}{t-s}\right)^{-2M} \|f_j(s, \cdot)\|_{L^2}^2 t^\beta ds dt. \end{aligned}$$

Exchanging the order of integration, and using the fact that  $t \sim t-s$  in the first part and that  $t \sim s$

in the second, we have the following.

$$\begin{aligned}
I_j^2 &\lesssim \sum_{k=1}^{\infty} 2^{-k} 2^{-2jmM} r^{-2M} \int_0^{2^{-k}r} \int_{2^k s}^{2^{k+1}s} t^{\beta+2M-1} \|f_j(s, \cdot)\|_{L^2}^2 dt ds \\
&\quad + \int_0^r \int_s^{2s} r(t-s)^{-2} \left(1 + \frac{2^{jm}r}{t-s}\right)^{-2M} \|f_j(s, \cdot)\|_{L^2}^2 s^{\beta} dt ds \\
&\lesssim \sum_{k=1}^{\infty} 2^{-k} 2^{-2jmM} \int_0^{2^{-k}r} (2^k s)^{\beta} \|f_j(s, \cdot)\|_{L^2}^2 ds + \int_0^r \int_1^{\infty} (1 + 2^{jm}\sigma)^{-2M} \|f_j(s, \cdot)\|_{L^2}^2 s^{\beta} d\sigma ds \\
&\lesssim 2^{-2jmM} \int_0^r \|f_j(s, \cdot)\|_{L^2}^2 s^{\beta} ds,
\end{aligned}$$

where we used  $\beta < 1$ . We thus have

$$I_j^2 \lesssim 2^{-2jmM} (2^j r^{\frac{1}{m}})^n \|f\|_{T^{\infty, 2, m}(t^{\beta} dt dy)}^2,$$

and the condition  $M > \frac{n}{2m}$  allows us to sum these estimates.  $\square$

*Remark 3.3.* Assuming off-diagonal estimates, instead of kernel estimates, allows to deal with differential operators  $L$  with rough coefficients. The harmonic analytic objects associated with  $L$  then fall outside the Calderón-Zygmund class, and it is common (see for instance [1]) for their boundedness range to be a proper subset of  $(1, \infty)$ . Here, our range  $(\frac{2n}{n+m(1-\beta)}, \infty]$  includes  $[2, \infty]$  as  $\beta < 1$ , which is consistent with [2]. In the case of classical tent spaces, i.e.,  $m = 1$  and  $\beta = -1$ , it is the range  $(2_*, \infty]$ , where  $2_*$  denotes the Sobolev exponent  $\frac{2n}{n+2}$ . We do not know, however, if this range is optimal.

*Remark 3.4.* Theorem 3.2 is a maximal regularity result for parabolic Carleson measure norms. This is quite natural from the point of view of non-linear parabolic PDE (where maximal regularity is often used), and such norm have, actually, already been used in the context of Navier-Stokes equations in [11], and, subsequently, for some geometric non-linear PDE in [12]. Theorem 3.1 is also reminiscent of Krylov's Littlewood-Paley estimates [13], and of their recent far-reaching generalization in [15]. In fact, the methods and results from [9], on which this paper relies, use the same circle of ideas (R-boundedness, Kalton-Weis  $\gamma$  multiplier theorem...) as [15]. The combination of these ideas into a “conical square function” approach to stochastic maximal regularity will be the subject of a forthcoming paper.

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